

Small time asymptotic on the diagonal for Hörmander's type hypoelliptic operators

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Abstract

We compute the small time asymptotic of the fundamental solution of Hörmander's type hypoelliptic operators with drift, at a stationary point, x_0 , of the drift field. We show that the order of the asymptotic depends on the controllability of an associated control problem and of its approximating system. If the control problem of the approximating system is controllable at x_0 , then so is also the original control problem, and in this case we show that the fundamental solution blows up as $t^{-N/2}$, where N is a number determined by the Lie algebra at x_0 of the fields, that define the hypoelliptic operator.

1 Introduction

In the following paper we will study the small time asymptotic on the diagonal of the fundamental solution to an hypoelliptic operator with drift. Let M be a n dimensional smooth manifold and let f_0, f_1, \dots, f_k be smooth vector fields on M , where f_0 is usually called *drift* field. We will consider the following operator on $C^\infty(\mathbb{R}^+ \times M)$

$$\frac{\partial \varphi}{\partial t} - f_0(\varphi) - \frac{1}{2} \sum_{i=1}^k f_i^2(\varphi) \quad \forall \varphi \in C^\infty(\mathbb{R}^+ \times M). \quad (1)$$

We will assume that the vector fields f_0, f_1, \dots, f_k satisfy Hörmander's condition of hypoellipticity (see [8]), that is

$$\text{Lie}_x \left\{ \frac{\partial}{\partial t} - f_0, f_1, \dots, f_k \right\} = \mathbb{R} \times T_x M \quad \forall x \in M. \quad (2)$$

Under this hypothesis the operator in (1) is hypoelliptic and, as shown for example in [9] and [8], it admits a well defined smooth fundamental solution, $p(t, x, y)$, for small time, that is given by the probability density of the stochastic process ξ_t at time t , starting at x at time 0 and driven by the stochastic differential equation

$$d\xi_t = f_0(\xi_t)dt + \sum_{i=1}^k f_i(\xi_t) \circ dw_i$$

where $w = (w_1, \dots, w_k)$ denotes a k -dimensional Brownian motion and \circ denotes the integration in the Stratonovich sense.

Given a stationary point, x_0 , of the drift field f_0 , i.e. $f_0(x_0) = 0$, the goal of this paper is to study the behaviour of the heat kernel, $p(t, x, y)$, for small time, in x_0 .¹

This problem has already been studied by many authors in the past and recent period. In particular, if the vector fields f_1, \dots, f_k are Lie bracket generating, i.e. they satisfy condition (2) without the contribution of f_0 , many results are already known. Indeed under this hypothesis it is well defined a distance function $d(x, y)$ determined by the fields f_1, \dots, f_k . Then it was proved by Léandre in [10] and [11] that

$$\lim_{t \rightarrow 0} t \log p(t, x, y) = -\frac{1}{2}d^2(x, y)$$

which generalizes a result by Varadhan [14] in the elliptic case. If moreover $f_0 \equiv 0$, then if $x \neq y$ and x and y are not conjugate along any minimal geodesic, then for some constant $C = C(x, y) \in \mathbb{R}$

$$p(t, x, y) = \frac{C + O(t)}{t^{n/2}} e^{-\frac{d^2(x, y)}{2t}},$$

as proved by Ben Arous [4] and Barilari, Boscain and Neel [3]. Here we see in particular that the order of the asymptotic in t is determined by the dimension of the manifold. On the other hand if $x = y$ and $f_0 \equiv 0$, Ben Arous and Léandre showed in [5] and [6] that

$$p(t, x, x) = \frac{C + O(\sqrt{t})}{t^{Q/2}} \quad (3)$$

where Q is the homogeneous dimension of the manifold and C is a constant depending on x . These two examples show that the order of degeneracy of the fundamental solution for small time can reveal interesting geometric properties and depends on the structure of the manifold.

In this paper we are going to determine the order of the asymptotic for small t and we will see that if f_0 is not identically zero the asymptotic (3) on the diagonal at a stationary point, x_0 , of the drift field could be more general, as it was already pointed out by Ben Arous and Léandre in [6]. The proof relies on a homogeneity property of the operator in (1) under the dilations. Indeed to understand better the behaviour of the fundamental solution $p(t, x_0, x_0)$, we will dilate the space around x_0 and rescale the time. We will see that, under a suitable dilation, this procedure splits the operator $\frac{\partial}{\partial t} - f_0 - \frac{1}{2} \sum_{i=1}^k f_i^2$ into a principal operator

$$\frac{\partial}{\partial t} - \hat{f}_0 - \frac{1}{2} \sum_{i=1}^k \hat{f}_i^2 \quad (4)$$

perturbed by a small operator that goes to zero as the space becomes larger. Here the fields $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k$ are nilpotent approximations of the original fields f_0, f_1, \dots, f_k around the point

¹The assumption of $f_0(x_0) = 0$ is important, but can be avoided in some cases. For example, one could be interested in studying the differential operator

$$\frac{\partial}{\partial t} - f_0 - \frac{1}{2} \sum_{i=1}^k (f_i^2 + (\operatorname{div}_\mu f_i) f_i)$$

that can be treated as an operator of the form (1)

$$\frac{\partial}{\partial t} - \tilde{f}_0 - \frac{1}{2} \sum_{i=1}^k f_i^2$$

once we take $\tilde{f}_0 = f_0 + \frac{1}{2} \sum_{i=1}^k (\operatorname{div}_\mu f_i) f_i$. The results of this paper apply also to an operator of this form and we will show the small time asymptotic of the fundamental solution on the diagonal at a point x_0 , where $f_0(x_0) = 0$ (not necessarily $\tilde{f}_0(x_0) = 0$). Details for this kind of operator are written in Remark 5.5.

x_0 . In this procedure it is very important to find a good coordinate system around x_0 , so that the dilation of the space produces an hypoelliptic principal operator (4). This is done by noting that the field f_0 acts in the operator just one time, while the fields f_1, \dots, f_k are applied twice. So we will give to the drift field a double weight with respect to the others. We explain this procedure in detail in Section 2.

We will prove that the asymptotic of the fundamental solution depends on the controllability of some associated control problems: consider the control problem

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^k u_i(t) f_i(x(t)) \quad (5)$$

where $u = (u_1, \dots, u_k) \in L^\infty(\mathbb{R}^+; \mathbb{R}^k)$ are the controls. If the control problem (5) is not controllable around the point x_0 , i.e. from x_0 we can not reach a neighborhood of x_0 using curves described by the control problem, then

$$p(t, x_0, x_0) = 0 \quad \forall t > 0.$$

We will consider also the control problem induced by the approximating system

$$\dot{x}(t) = \hat{f}_0(x(t)) + \sum_{i=1}^k u_i(t) \hat{f}_i(x(t)). \quad (6)$$

If it is controllable, then so is also the original one (5), and, if we denote by $q_0(t, x, y)$ the fundamental solution of the approximating hypoelliptic operator in (4), we prove in Theorem 5.4 that

$$p(t, x_0, x_0) = \frac{q_0(1, x_0, x_0) + O(t)}{t^{N/2}}, \quad (7)$$

where the factor $t^{N/2}$ comes from the change of coordinates given by the dilations and it is determined by the order of homogeneity under the dilations of a volume form like $dx_1 \wedge \dots \wedge dx_n$ around x_0 .

The number N is an integer depending on the structure of the Lie algebra generated by the fields f_0, f_1, \dots, f_k in x_0 . It is computed in detail in Section 4, but to anticipate here how it is found, we give a brief explanation of it: for $j \in \mathbb{N}$, let

$$d_j := \dim \operatorname{span}_{x_0} \{ [f_{i_1}, [\dots, [f_{i_{l-1}}, f_{i_l}] \dots]] \text{ such that } \#\{i_h > 0\} + 2\#\{i_h = 0\} \leq j \}$$

be the dimension of the space generated by all the Lie brackets at x_0 of at most j vector fields f_0, f_1, \dots, f_k , where the field f_0 is counted two times. Then the number N is defined as

$$N = \sum_{j \geq 1} j \cdot d_j.$$

For an operator coming from a sub-Riemannian manifold with no drift field, N is equal to the homogeneous dimension \mathcal{Q} of the manifold. If the drift field is not zero, then N could be more general.

In the intermediate case in which the approximating control problem (6) is not controllable in x_0 , but the original control problem (5) is still controllable, then the behaviour of the asymptotic can be more general. It could vanish, or blow up faster than $t^{-N/2}$ and even exponentially fast, as it was already pointed out by Ben Arous and Léandre in [6].

The paper is organized as follows. We will begin by describing in details in Section 2 the homogeneity properties of the operator in (1). In particular we will derive the conditions that

the dilations have to satisfy in order to produce the right split of the dilated operator, into a hypoelliptic principal part, perturbed by a small operator. We will also give a brief introduction into Duhamel's formula in Subsection 2.1, which will be an important tool to study the perturbed operator. In Section 3 we will introduce the coordinates that give the right dilations of the space. This coordinates are defined from a filtration of the tangent space to x_0 determined by the fields f_0, f_1, \dots, f_k and give rise to a graded structure around x_0 , which defines an anisotropic dilation. In Section 4 we will define the nilpotent approximation, that determines the principal part of the vector fields f_0, f_1, \dots, f_k , which define the principal operator (4). We will compute also the integer N appearing in the asymptotic (7), that derives from the change of the volume form under the dilations. In Section 5 we will prove the asymptotic (7), by using the tools introduced in the previous sections. In the following Section 6 we will focus our study on the operator derived from the nilpotent approximating system and its associated control problem. By proving a modification of Stroock and Varadhan's support theorem, we will give a necessary and sufficient condition for the positivity of the fundamental solution of the principal operator, that is based on the controllability of the approximating control system. Finally we will end the paper with Section 7, where we show a series of examples, illustrating how this formula recovers in particular the known results recalled in the introduction.

2 The fundamental solution and its behavior under the action of a dilation

Let M be an orientable n -dimensional manifold (without boundary) and let μ be a volume form on M . Given f_0, f_1, \dots, f_k smooth vector fields on M we consider the following partial differential operator:

$$\frac{\partial \varphi}{\partial t} - f_0(\varphi) - \frac{1}{2} \sum_{i=1}^k f_i^2(\varphi) \quad \forall \varphi \in C^\infty(\mathbb{R} \times M). \quad (8)$$

We will call f_0 the *drift* field and we will denote by L the operator $f_0 + \frac{1}{2} \sum_{i=1}^k f_i^2$. Let us recall the definition of fundamental solution.

Definition 2.1. *The fundamental solution of an operator $\frac{\partial}{\partial t} - \mathcal{L}$ over $\mathbb{R} \times M$ with respect to the volume μ is a function $p(t, x, y) \in C^\infty(\mathbb{R}^+ \times M \times M)$ such that*

- *for every fixed $y \in M$, it holds $\frac{\partial}{\partial t} p(t, x, y) = \mathcal{L}_x p(t, x, y)$, where the operator \mathcal{L} acts on the x variable;*
- *for any $\varphi_0 \in C_0^\infty(M)$, we have*

$$\lim_{t \searrow 0} \int_M p(t, x, y) \varphi_0(y) \mu(y) = \varphi_0(x) \quad \text{uniformly in } t.$$

In other words, if we want to solve the partial differential equation $\frac{\partial \varphi}{\partial t} = \mathcal{L} \varphi$ with initial condition $\varphi(0, x) = \varphi_0(x)$, the fundamental solution allows to reconstruct φ by convolution of φ_0 with $p(t, x, y)$.

Remark 2.2. The fundamental solution depends on the given volume μ in the following way: let μ and ν be two volume forms on M , and let g be a smooth function such that $\nu = e^g \mu$. Let

p_μ and p_ν denote the fundamental solutions of $\frac{\partial}{\partial t} - \mathcal{L}$ with respect to μ and ν respectively. Then for every initial condition $\varphi_0 \in C_0^\infty(M)$, the solution $\varphi(t, x)$ of

$$\begin{cases} \frac{\partial \varphi}{\partial t} = f_0(\varphi) + \frac{1}{2} \sum_{i=1}^k f_i^2(\varphi) \\ \varphi(0, x) = \varphi_0(x) \end{cases} \quad (9)$$

is given by

$$\begin{aligned} \varphi(t, x) &= \int_M p_\mu(t, x, y) \varphi_0(y) \mu(y) \\ &= \int_M p_\nu(t, x, y) \varphi_0(y) \nu(y) = \int_M p_\nu(t, x, y) \varphi_0(y) e^{g(y)} \mu(y), \end{aligned}$$

where the equalities follow since the solution is unique for smooth vector fields. Since φ_0 is arbitrary, we have

$$p_\mu(t, x, y) = e^{g(y)} p_\nu(t, x, y) \quad \forall t > 0, \forall x, y \in M.$$

From the point of view of the asymptotic of the fundamental solution on the diagonal, it follows that the two asymptotics are the same for both volume forms up to a multiplicative constant $e^{g(x_0)} \neq 0$ depending on the relation between the two volume forms and on the point where we compute the asymptotic. For the study of the small time asymptotic on the diagonal, we will then suppose $\mu = dx_1 \wedge \dots \wedge dx_n$ near the point x_0 .

The existence of a fundamental solution for an operator $\frac{\partial}{\partial t} - \mathcal{L}$ is not always guaranteed, but in 1967 Hörmander published an important paper on the properties of an operator like the one in (8). An operator \mathcal{L} is called *hypoelliptic* if whenever, for a function φ , $\mathcal{L}\varphi \in C^\infty(U)$ on an open set U , then $\varphi \in C^\infty(U)$. In [8] Hörmander proved a condition of hypoellipticity for operators of the form (8), namely if

$$\text{Lie}_x \left\{ \frac{\partial}{\partial t} - f_0, f_1, \dots, f_k \right\} = \mathbb{R} \times T_x M \quad \forall x \in M,$$

then the operator $\frac{\partial}{\partial t} - L$ is hypoelliptic. Throughout this paper we will always assume that the fields f_0, f_1, \dots, f_k satisfy Hörmander condition.

Under this assumption of hypoellipticity, the operator in (8) admits a fundamental solution, $p(t, x, y)$ (see the backward Kolmogorov equation and [9], Chapter 4.6 for the study on \mathbb{R}^n and Chapter 5 for the equation on a manifold). It is given by the probability density function of the process ξ_t that satisfies the following stochastic differential equation in Stratonovich form

$$\begin{cases} d\xi_t = f_0(\xi_t)dt + \sum_{i=1}^k f_i(\xi_t) \circ dw_i, \\ \xi_0 = x \end{cases} \quad (10)$$

where ξ_t is a process on M and (w_1, \dots, w_k) denotes a k -dimensional Brownian motion. More explicitly, the fundamental solution of (8) with respect to a volume form μ on M is the function $p(t, x, y)$ such that for every measurable set $A \subset M$, the probability of the process ξ_t to be in A at time t is given by

$$P[\xi_t \in A | \xi_0 = x] = \int_A p(t, x, y) \mu(y).$$

Moreover, for every initial condition $\varphi_0(x)$, the corresponding solution of the differential equation is

$$\varphi(t, x) = \int_M p(t, x, y) \varphi_0(y) \mu(y) = \mathbb{E}[\varphi_0(\xi_t) | \xi_0 = x],$$

that is the expectation value of $\varphi_0(\xi_t)$ at time t , knowing that $\xi_0 = x$.

Let x_0 be a fixed point of the drift field, $f_0(x_0) = 0$, our goal in this paper is to understand the small time behaviour of the fundamental solution at the point x_0 . Our method will proceed as follows: we will dilate the space around the point x_0 , and we will rescale the time accordingly, so that the space around x_0 will be magnified and we can study better the behaviour of the solution at x_0 .

Definition 2.3. Let (U, x) be any coordinate neighborhood of x_0 and let (w_1, \dots, w_n) be positive integers, that we will call weights of the coordinates (x_1, \dots, x_n) .

For $\epsilon > 0$ we define the dilation, δ_ϵ , of order ϵ and weights (w_1, \dots, w_n) around x_0 as the function $\delta_\epsilon : U \rightarrow U$, such that

$$\delta(x_1, \dots, x_n) := (\epsilon^{w_1} x_1, \epsilon^{w_2} x_2, \dots, \epsilon^{w_n} x_n) \quad \forall x = (x_1, \dots, x_n) \in U. \quad (11)$$

For $\epsilon > 1$ we understand the dilation defined actually on a smaller neighborhood of x_0 in U .

Under the action of the dilations δ_ϵ , the coordinate functions and the coordinate vector fields behave as

$$x_i \circ \delta_\epsilon = \epsilon^{w_i} x_i, \quad \delta_{\frac{1}{\epsilon}*} \frac{\partial}{\partial x_i} = \frac{1}{\epsilon^{w_i}} \frac{\partial}{\partial x_i} \quad \forall 1 \leq i \leq n. \quad (12)$$

Let the volume μ be represented in the coordinate neighborhood (U, x) by $\mu = dx_1 \wedge \dots \wedge dx_n$. By Remark 2.2 this assumption is not restrictive for the study of the asymptotic along the diagonal. Then the volume form μ changes under the action of the dilation δ_ϵ as

$$\delta_\epsilon^*(dx_1 \wedge \dots \wedge dx_n) = \epsilon^{\sum_{i=1}^n w_i} dx_1 \wedge \dots \wedge dx_n. \quad (13)$$

When we apply a dilation to the space around x_0 and we rescale the time variable, also the fundamental solution is changed accordingly, as it is proved in the following proposition.

Proposition 2.4. Let (U, x) be a coordinate neighborhood around the point x_0 and let μ be a volume form on M such that $\mu = dx_1 \wedge \dots \wedge dx_n$ in U . For weights (w_1, \dots, w_n) and $0 < \epsilon < 1$ consider the dilation $\delta_\epsilon : U \rightarrow U$. Let $p(t, x, y)$ be the fundamental solution of the operator in (8) with respect to the volume μ . Then the fundamental solution on U of the operator

$$\frac{\partial}{\partial t} - \epsilon^a \left(\delta_{1/\epsilon*} f_0 + \frac{1}{2} \sum_{i=1}^k (\delta_{1/\epsilon*} f_i)^2 \right) \quad (14)$$

is the function

$$q_\epsilon(t, x, y) := \epsilon^{\sum_{i=1}^n w_i} p(\epsilon^a t, \delta_\epsilon(x), \delta_\epsilon(y)) \quad \forall x, y \in U. \quad (15)$$

Here $\delta_{1/\epsilon*} X$ denotes the pushforward of a vector field X under the action of δ_ϵ and a is a real positive number.

Remark 2.5. The coefficient of normalization $\epsilon^{\sum_{i=1}^n w_i}$, that we have used to define q_ϵ , is necessary in order to reconstruct all the solutions of the differential operator, by convolution with q_ϵ . This coefficient appears as soon as we make a change of coordinates in the integral of the convolution. Moreover, we will see that from this coefficient we find the order of the asymptotic of the fundamental solution for small time.

Proof. First of all notice that the dilation $\delta_{1/\epsilon} : \delta_\epsilon(U) \rightarrow U$ can be defined only on the smaller neighborhood $\delta_\epsilon(U)$ of U . Then the fields $\delta_{1/\epsilon*} f_i$ are vector fields just on the coordinate neighborhood U .

Next let us prove the first property of the fundamental solution, i.e. that the function q_ϵ is a solution of the operator in (14). For convenience, we call ψ the dilation from $\mathbb{R}^+ \times \delta_{1/\epsilon}U$ to $\mathbb{R}^+ \times U$ defined by $\psi(t, x) := (\epsilon^a t, \delta_\epsilon(x))$. Then the function q_ϵ can be written as $q_\epsilon(t, x, y) = p(\psi(t, x), \delta_\epsilon(y))$ and the operator in (14) is

$$\psi_*^{-1} \frac{\partial}{\partial t} - \psi_*^{-1} f_0 - \frac{1}{2} \sum_{i=1}^k (\psi_*^{-1} f_i)^2,$$

where we are using a little abuse of notation, by considering $\frac{\partial}{\partial t}, f_0, \dots, f_k$ as vector fields defined on the product space $\mathbb{R}^+ \times U$. Recall the definition of pushforward of a vector field X under the action of a diffeomorphism φ : for every function g we have $\varphi_{x*}(X)(f)|_{\varphi(x)} = X_x(f \circ \varphi|_x)$. Then we compute for fixed $y \in U$

$$\begin{aligned} \left(\psi_*^{-1} \frac{\partial}{\partial t} \right) q_\epsilon(t, x, y) &= \epsilon^{\sum_{i=1}^n w_i} \left(\psi_*^{-1} \frac{\partial}{\partial t} \right) \Big|_{(t, x, y)} (p(\psi(t, x), \delta_\epsilon y)) \\ &= \epsilon^{\sum_{i=1}^n w_i} \psi_* \left(\left(\psi_*^{-1} \frac{\partial}{\partial t} \right) \Big|_{(t, x, y)} \right) p|_{(\psi(t, x), \delta_\epsilon y)} \\ &= \epsilon^{\sum_{i=1}^n w_i} \frac{\partial}{\partial t} \Big|_{(\psi(t, x), \delta_\epsilon y)} p|_{(\psi(t, x), \delta_\epsilon y)} \\ &= \epsilon^{\sum_{i=1}^n w_i} L|_{(\psi(t, x), y)} p|_{(\psi(t, x), \delta_\epsilon y)} \end{aligned}$$

where the last equality follows since p is the fundamental solution of the operator $\frac{\partial}{\partial t} - L$. Applying the same computations to the vector fields $\psi_*^{-1} f_i$ for $i = 0, \dots, k$, we find that q_ϵ satisfies

$$\psi_*^{-1} \frac{\partial}{\partial t} q_\epsilon(t, x, y) = \psi_*^{-1} f_0(q_\epsilon(t, x, y)) + \frac{1}{2} \sum_{i=1}^k (\psi_*^{-1} f_i)^2(q_\epsilon(t, x, y))$$

and hence q_ϵ is a solution for the operator in (14).

Let us prove the second property of a fundamental solution. Here we will see that the constant of normalization $\epsilon^{\sum_{i=1}^n w_i}$ in the definition of q_ϵ is exactly the parameter that we need in order to construct the other solutions of the partial differential equation by convolution with the fundamental solution. Indeed let us prove that for any $\varphi_0 \in C_0^\infty(U)$, it holds

$$\lim_{t \searrow 0} \int_U q_\epsilon(t, x, y) \varphi_0(y) \mu(y) = \varphi_0(x).$$

This follows by a change of variable and the same property valid for the fundamental solution $p(t, x, y)$:

$$\begin{aligned} \lim_{t \searrow 0} \int_U q_\epsilon(t, x, y) \varphi_0(y) \mu(y) &= \lim_{t \searrow 0} \int_U \epsilon^{\sum_{i=1}^n w_i} p(\epsilon^a t, \delta_\epsilon x, \delta_\epsilon y) \varphi_0(y) \mu(y) \\ &= \lim_{t \searrow 0} \int_M \epsilon^{\sum_{i=1}^n w_i} p(\epsilon^a t, \delta_\epsilon x, \delta_\epsilon y) \varphi_0(y) \mu(y) \end{aligned}$$

We can integrate on M , because, since φ_0 has compact support in U , we can extend φ_0 to be zero outside U . Now let us do a change of variable with $z = \delta_\epsilon y$. As computed in (13), the volume is transformed as $\mu(z) = \epsilon^{\sum_{i=1}^n w_i} \mu(y)$. Then

$$\begin{aligned} \lim_{t \searrow 0} \int_U q_\epsilon(t, x, y) \varphi_0(y) \mu(y) &= \lim_{t \searrow 0} \int_M \epsilon^{\sum_{i=1}^n w_i} p(\epsilon^a t, \delta_\epsilon x, z) (\varphi_0 \circ \delta_{1/\epsilon})(z) \frac{\mu(z)}{\epsilon^{\sum_{i=1}^n w_i}} \\ &= (\varphi_0 \circ \delta_{1/\epsilon})(\delta_\epsilon x) = \varphi_0(x) \end{aligned}$$

where the second equality follows because p is a fundamental solution. \square

Let us investigate better the behaviour of the fields f_i under the action of the dilations. Let us write every component $f_i^{(j)}$ of the fields f_i in a Taylor expansion centered in $x_0 = 0$ for x in the coordinate neighborhood U

$$f_i^{(j)}(x) \frac{\partial}{\partial x_j} = f_i^{(j)}(0) \frac{\partial}{\partial x_j} + \sum_{l=1}^n \frac{\partial f_i^{(j)}(0)}{\partial x_l} x_l \frac{\partial}{\partial x_j} + o(x).$$

By the properties of a dilation acting on the coordinate functions and on the coordinate vector fields, (12), when we apply a dilation to the vector fields f_i , every component has a different degree with respect to ϵ . Depending on the value of the weights (w_1, \dots, w_n) and on the coefficients of the Taylor expansion of the fields f_i , for every $i = 0, \dots, k$, there exist an integer α_i and a principal vector field \hat{f}_i such that

$$\delta_{1/\epsilon*} f_i = \frac{1}{\epsilon^{\alpha_i}} \hat{f}_i + o\left(\frac{1}{\epsilon^{\alpha_i}}\right),$$

where \hat{f}_i contains the components of every $f_i^{(j)} \frac{\partial}{\partial x_j}$ that is homogeneous of degree $-\alpha_i$ with respect to the dilations. Applying this formula to the dilated operator in (14), we find that the operator L rescales as

$$\delta_{1/\epsilon*} f_0 + \frac{1}{2} \sum_{i=1}^k (\delta_{1/\epsilon*} f_i)^2 = \frac{1}{\epsilon^{\alpha_0}} \hat{f}_0 + \frac{1}{2} \sum_{i=1}^k \frac{1}{\epsilon^{2\alpha_i}} \hat{f}_i^2 + o\left(\frac{1}{\epsilon^\alpha}\right)$$

where $\alpha := \max\{\alpha_0, 2\alpha_1, \dots, 2\alpha_k\}$.

The main task in our study will be to find suitable coordinates and good weights w_i , so that all the principal parts of the vector field f_0 and of f_1^2, \dots, f_k^2 rescale with the same degree under the dilations. In this way the principal part in the dilated operator is homogeneous of order $-\alpha$ and if we choose $a = \alpha$, the operator L in (14) can be written as

$$\epsilon^\alpha \left(\delta_{1/\epsilon*} f_0 + \frac{1}{2} \sum_{i=1}^k (\delta_{1/\epsilon*} f_i)^2 \right) = \hat{f}_0 + \frac{1}{2} \sum_{i=1}^k \hat{f}_i^2 + \mathcal{X}_\epsilon. \quad (16)$$

where \mathcal{X}_ϵ is a differential operator that goes to 0 as ϵ goes to zero. As proved in Proposition 2.4, its fundamental solution on U is given by

$$q_\epsilon(t, x, y) := \epsilon^{\sum_{i=1}^n w_i} p(\epsilon^\alpha t, \delta_\epsilon x, \delta_\epsilon y). \quad (17)$$

To an operator like the one in (16) we can apply Duhamel's formula, that gives the asymptotic of the fundamental solution as a perturbation of the asymptotic of the fundamental solution of the principal operator.

2.1 Duhamel's formula

In this section we recall briefly a famous formula, called Duhamel's formula, which allows to find the asymptotic of the fundamental solution of a perturbed operator, once we have the explicit fundamental solution of the principal part of the operator. This method is presented, among others, in Chapter 3 of [12] and in [2].

Let \mathcal{L} be an operator on a Hilbert space with fundamental solution $p(t, x, y)$ and let us define the following operator on the Hilbert space

$$e^{t\mathcal{L}} \varphi(x) = \int p(t, x, y) \varphi(y) dy.$$

By the properties of the fundamental solution this is an heat operator $e^{t\mathcal{L}}$, i.e. an operator such that

$$\frac{\partial e^{t\mathcal{L}}\varphi}{\partial t} = \mathcal{L}e^{t\mathcal{L}}\varphi \quad \text{and} \quad \lim_{t \rightarrow 0} e^{t\mathcal{L}}\varphi = \varphi,$$

Suppose that \mathcal{L} can be decomposed in a sum,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{X},$$

of a principal part, \mathcal{L}_0 , and a perturbation, \mathcal{X} , and assume that \mathcal{L}_0 has a well defined heat operator $e^{t\mathcal{L}_0}$. Then Duhamel's formula allows to reconstruct the heat operator of \mathcal{L} by a perturbation of the heat operator of \mathcal{L}_0 (see Chapter 3 of [12] for a proof), namely

$$e^{t\mathcal{L}} = e^{t\mathcal{L}_0} + \int_0^t e^{(t-s)\mathcal{L}} \mathcal{X} e^{s\mathcal{L}_0} ds = e^{t\mathcal{L}_0} + e^{t\mathcal{L}} * \mathcal{X} e^{t\mathcal{L}_0} \quad (18)$$

where with $*$ we denote the convolution operator between two operators, $A(t)$ and $B(t)$, on the Hilbert space:

$$(A * B)(t) = \int_0^t A(t-s)B(s)ds.$$

Let $a(t, x, y)$ and $b(t, x, y)$ be the heat kernels of $A(t)$ and $B(t)$ respectively and let X be an operator. Then the kernel of $(A * XB)(t)$ is obtained as follows: let φ be a function in the Hilbert space, then

$$\begin{aligned} [(A * XB)(t)\varphi](x) &= \left[\int_0^t A(t-s)XB(s)ds \varphi \right](x) \\ &= \left[\int_0^t A(t-s) \left[X \int_M b(s, \cdot, y)\varphi(y)dy \right] ds \right](x) \\ &= \int_0^t \int_M a(t-s, x, z)X_z \left(\int_M b(s, z, y)\varphi(y)dy \right) dz ds \\ &= \int_M \left(\int_0^t \int_M a(t-s, x, z)X_z b(s, z, y) dz ds \right) \varphi(y) dy \end{aligned}$$

so the heat kernel of $(A * XB)(t)$ is

$$(a * Xb)(t, x, y) = \int_0^t \int_M a(s, x, z)X_z b(t-s, z, y) dz ds.$$

From (18) we can now derive an approximation of the heat kernel $p(t, x, y)$ of the perturbed operator \mathcal{L} , by means of the heat kernel $p_0(t, x, y)$ of the principal operator:

$$p(t, x, y) = p_0(t, x, y) + (p * \mathcal{X}p_0)(t, x, y). \quad (19)$$

We will make use of this formula in the next section to find the asymptotic expansion of the fundamental solution of (1).

2.2 The perturbative method

We can apply Duhamel's formula to the operator in (16). Indeed it is the sum of a principal operator

$$L_0 = \hat{f}_0 + \frac{1}{2} \sum_{i=1}^k \hat{f}_i^2$$

perturbed by a small operator \mathcal{X}_ϵ . If we find good coordinates and weights, so that $\frac{\partial}{\partial t} - L_0$ admits a fundamental solution $q_0(t, x, y)$, then by Duhamel's formula (19) the asymptotic of the fundamental solution q_ϵ is

$$q_\epsilon(t, x, y) = q_0(t, x, y) + (q_\epsilon * \mathcal{X}_\epsilon q_0)(t, x, y). \quad (20)$$

From here we will immediately conclude our study, indeed let us choose in the definition of q_ϵ , (17), $x = x_0$, let ϵ go to zero as $\epsilon = t^{1/\alpha}$ and fix the time variable $t = 1$, then we get

$$p(t, x_0, x_0) = \frac{q_\epsilon(1, x_0, x_0)}{\epsilon \sum_{i=1}^n w_i} = \frac{1}{\epsilon \sum_{i=1}^n w_i} (q_0(1, x_0, x_0) + (q_\epsilon * \mathcal{X}_\epsilon q_0)(1, x_0, x_0)).$$

In conclusion, in the choice of the coordinates (U, x) and the weights (w_1, \dots, w_n) it will be important that the principal parts, $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k$, of the vector fields make homogeneous the principal part of the dilated operator and, moreover, that they satisfy Hörmander condition, so that it is guaranteed the existence of a fundamental solution q_0 of the principal operator L_0 . Finally we will need to check that the remainder term $(q_\epsilon * \mathcal{X}_\epsilon q_0)(1, x_0, x_0)$ in the asymptotic of p goes to zero, as t (and therefore ϵ) goes to zero.

3 Graded structure induced by a filtration

In this section we will introduce some notation and recall the definition of local graded structure of a manifold, induced by a filtration. This terminology will be essential to find the right coordinate to rescale the differential operator L and to compute the order of the asymptotic of the fundamental solution. We will constantly refer to Bianchini and Stefani's paper [7].

3.1 Chart adapted to a filtration

Let M be a n dimensional smooth manifold and f_0, f_1, \dots, f_k be smooth vector fields on M , that satisfy Hörmander condition (2). We will consider the hypoelliptic operator on $\mathbb{R} \times M$ defined as

$$\frac{\partial}{\partial t} - f_0 - \sum_{i=1}^k f_i^2.$$

As noticed in the previous section the role played by the drift field f_0 and the other vector fields, $\{f_1, \dots, f_k\}$, in the sum of squares, is different, and in particular the fields $\{f_1, \dots, f_k\}$ are applied twice as many times as the drift field is. For this reason we want to treat differently the two kinds of fields, by giving to them two different *weights*.

Let $\text{Lie}X$ be the Lie algebra on \mathbb{R} generated by a set $\{X_0, X_1, \dots, X_k\}$ of noncommutative indeterminates.

Definition 3.1. For every bracket Λ in $\text{Lie}X$ we denote by $|\Lambda|_i$ the number of times that the indeterminate X_i appears in the definition of Λ . We will call this number the length of Λ with respect to X_i .

For example, the bracket $\Lambda = [X_0, [X_2, X_0]]$ has lengths $|\Lambda|_0 = 2$, $|\Lambda|_1 = 0$ and $|\Lambda|_2 = 1$, and it has zero length with respect to any other indeterminate.

By fixing a weight, l_i , to every indeterminate X_0, \dots, X_k we can define the weight of a bracket Λ .

Definition 3.2. Given a set of integers (l_0, l_1, \dots, l_k) , we define the weight of a bracket $\Lambda \in \text{Lie}X$ as

$$||\Lambda|| := \sum_{i=0}^k l_i |\Lambda|_i \quad \text{if } \Lambda \neq 0$$

and we set $||0|| = 0$.

In order to give different importance to the drift field, with respect to the other vector fields, in the following we will fix the integers to be

$$l_0 = 2 \quad \text{and} \quad l_1 = \dots = l_k = 1. \quad (21)$$

This means that the indeterminate X_0 will have weight 2, while the other indeterminates will have weight 1. For more complex Lie brackets, we have for example that the weight of the bracket considered before $\Lambda = [X_0, [X_2, X_0]]$ is

$$||\Lambda|| = 2 \cdot |\Lambda|_0 + 1 \cdot |\Lambda|_2 = 5.$$

By means of the weight of the indeterminates we introduce now a filtration of the Lie algebra spanned by f_0, f_1, \dots, f_k in the following way. For every bracket Λ in $\text{Lie}X$ we denote by Λ_f the vector field on M obtained by replacing every indeterminate X_i with the corresponding field f_i for $0 \leq i \leq k$. Then we define an increasing filtration $\mathcal{L} = \{L_i\}_{i \geq 0}$ of $\text{Vec}(M)$ by

$$L_i = \text{span}\{\Lambda_f : \Lambda \in \text{Lie}X, ||\Lambda|| \leq i\}. \quad (22)$$

In other words L_i is the subalgebra of $\text{Vec}(M)$ that contains all the vector fields obtained from a bracket of weight less than or equal to i . In particular, following our choice of weights, the first subspaces of the filtration are

$$\begin{aligned} L_0 &= \{0\} \\ L_1 &= \text{span}\{f_1, \dots, f_k\} \\ L_2 &= \text{span}\{f_i, [f_i, f_j], f_0 : i, j = 1, \dots, k\} \\ L_3 &= \text{span}\{f_i, [f_i, f_j], f_0, [f_i, [f_j, f_h]], [f_0, f_i] : i, j, h = 1, \dots, k\} \\ &\vdots \end{aligned}$$

Notice moreover that for every $i, j \geq 0$, the following properties hold

- $L_i \subset L_{i+1}$
- $[L_i, L_j] \subset L_{i+j}$
- $\bigcup_{i \geq 0} L_i = \text{Lie}\{f_0, f_1, \dots, f_k\}$ and $\bigcup_{i \geq 0} L_i(x) = T_x M$, for every $x \in M$, since by assumption the family $\{f_0, \dots, f_k\}$ satisfies the weak Hörmander condition (2).

When we evaluate \mathcal{L} at the stationary point x_0 we get a stratification of the tangent space $T_{x_0} M$ at x_0 and this stratification will induce a very peculiar choice of coordinates around x_0 .

Let

$$d_i := \dim L_i(x_0) \quad \forall i \geq 0.$$

In particular, $d_0 = 0$ and $d_1 \leq k$. Moreover, by Hörmander condition (2), there exists a smallest integer m such that $L_m(x_0) = T_{x_0} M$. We will call this number the *step* of the filtration \mathcal{L} at x_0 .

The filtration \mathcal{L} induces a particular choice of coordinates centered at x_0 , as proved by the following proposition.

Proposition 3.3 (Bianchini, Stefani [7]). *There exists a chart (U, x) centered at x_0 such that for every $1 \leq j \leq m$*

$$(i) \quad L_j(x_0) = \text{span}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{d_j}}\right\}$$

(ii) $Dx_h(x_0) = 0$ for every differential operator $D \in \mathcal{A}^j := \{Z_1 \cdots Z_l \text{ with } Z_s \in L_{i_s} \text{ and } i_1 + \cdots + i_l \leq j\}$ and for every $h > d_j$.

Definition 3.4. *We will call a chart that satisfies the properties of Proposition 3.3 an adapted chart to the filtration \mathcal{L} at x_0 .*

Since this kind of coordinates will reveal to be very important in our study, we give here the proof of the proposition, which relies upon the following Lemma:

Lemma 3.5. *Let m be the step of the filtration \mathcal{L} at x_0 and let $j < m$ be an integer. Let $\varphi \in C^\infty(M)$ be such that $d_{x_0}\varphi \neq 0$ and $Z \cdot \varphi(x_0) = 0$ for all $Z \in L_j$. Then there exists an open neighborhood U of x_0 and a function $\hat{\varphi} \in C^\infty(U)$ such that*

$$d_{x_0}\varphi = d_{x_0}\hat{\varphi} \quad \text{and} \quad D \cdot \hat{\varphi}(x_0) = 0 \quad \forall D \in \mathcal{A}^j = \{Z_1 \cdots Z_l : Z_s \in L_{i_s}, i_1 + \cdots + i_l \leq j\}.$$

Proof. Let Y_1, \dots, Y_n be vector fields on M such that they form a basis of $T_{x_0}M$ at x_0 and such that

- $\{Y_1, \dots, Y_{d_i}\}$ are in L_i and form a basis of L_i at x_0 , for every $i \leq j$,
- $Y_i \cdot \varphi(x_0) = 0$ for every $i \leq n-1$,
- $Y_n \cdot \varphi(x_0) = 1$.

We choose the chart $y = (y_1, \dots, y_n)$ as the local inverse of the map

$$(y_1, \dots, y_n) \mapsto e^{y_n Y_n} \cdots e^{y_1 Y_1} x_0.$$

Then the function $\hat{\varphi} := y_n$ is such that $d_{x_0}\varphi = d_{x_0}\hat{\varphi}$.

Let $D = Z_1 \cdots Z_l \in \mathcal{A}^j$ with $Z_s \in L_{i_s}$ and $i_1 + \cdots + i_l \leq j$ and let us prove the second property required for the function $\hat{\varphi}$, by induction on l . Since $d_{x_0}\varphi = d_{x_0}\hat{\varphi}$ and $Z \cdot \varphi(x_0) = 0$ for all $Z \in L_j$ by hypothesis, the property is satisfied for $l = 1$. For $l > 1$, since $Z_l \in L_{i_l}$ we can write $Z_l(x_0) = \sum_{i=1}^{d_{i_l}} a_i Y_i(x_0)$ for some a_i so

$$D \cdot \hat{\varphi}(x_0) = \sum_{i=1}^{d_{i_l}} a_i (Z_{l-1} \cdot Y_i \cdot Z_{l-2} \cdots Z_1 + [Y_i, Z_{l-1}] \cdot Z_{l-2} \cdots Z_1) \cdot \hat{\varphi}(x_0)$$

The second component on the left side vanishes because, by the definition of the filtration, $[L_i, L_h] \in L_{i+h}$, so we can apply on this component the induction hypothesis. By applying again the same commutation we have

$$D \cdot \hat{\varphi}(x_0) = \sum_{i=1}^{d_{i_l}} a_i Z_{l-1} \cdots Z_1 \cdot Y_i \cdot \hat{\varphi}(x_0).$$

Iterating the same procedure also to Z_{l-1}, \dots, Z_1 we can write $D \cdot \hat{\varphi}(x_0)$ as a linear combination of elements of the type

$$Y_{i_l} \cdots Y_{i_1} \cdot \hat{\varphi}(x_0),$$

with $1 \leq i_h \leq d_j < n$ for every $h = 1, \dots, l$. Therefore we get $D \cdot \hat{\varphi}(x_0) = D \cdot y_n(x_0) = 0$. \square

Proof of Proposition 3.3. Let (U, x) be any chart centered at x_0 . We can get a chart with property (i) of the proposition by a linear change of coordinates. Let us still denote it by (U, x) . For every $i \leq n$ let j be such that $d_j < i \leq d_{j+1}$. Then the coordinate function x_i satisfies the hypothesis of the Lemma with respect to the integer j . By applying the Lemma to each function of the chart we get the statement. \square

We will see an example of an adapted chart in Section 4.1.

3.2 Graded structure

For $1 \leq i \leq m$ we define the integers

$$k_i := d_i - d_{i-1},$$

which indicate the number of new coordinates achieved with every new layer of the filtration at x_0 .

Definition 3.6. Let us denote a point $x = (x_1, \dots, x_n) \in U$ by the m -tuple $(x^1, x^2, \dots, x^m) \in \mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2} \oplus \dots \oplus \mathbb{R}^{k_m}$, where each component $x^i := (x_{d_{i-1}+1}, \dots, x_{d_i})$ is a vector of length k_i . Then for every $\epsilon > 0$ we define the anisotropic dilations around x_0 of factor ϵ as

$$\delta_\epsilon(x) = \delta_\epsilon(x^1, \dots, x^m) := (\epsilon x^1, \epsilon^2 x^2, \dots, \epsilon^m x^m).$$

Notice that if $\epsilon > 1$, the map δ_ϵ is actually defined only in a smaller neighborhood of x_0 , but we will still write the map $\delta_\epsilon : U \rightarrow U$, understanding "locally defined".

The dilations δ_ϵ act on every coordinate function and on the coordinate vector fields with a different weight, namely

$$x_j \circ \delta_\epsilon = \epsilon^i x_j, \quad (\delta_\epsilon)_* \frac{\partial}{\partial x_j} = \epsilon^i \frac{\partial}{\partial x_j} \quad \forall d_{i-1} < j \leq d_i.$$

For every $1 \leq i \leq n$ let w_j be the order of expansion of the coordinate x_j , that is $w_j := i$ if $d_{i-1} < j \leq d_i$. We call w_j the *weight* of the coordinate x_j . Then the dilation δ_ϵ is a particular choice of the dilations defined in Definition 2.3 with respect to the coordinates induced by the filtration \mathcal{L} and the weights (w_1, \dots, w_n) .

Accordingly we define the *weight* of a monomial to be

$$\mathcal{W}(x_1^{\alpha_1} \dots x_n^{\alpha_n}) := \sum_{j=1}^n \alpha_j w_j$$

and the weight of a polynomial to be the greatest order of its monomials. Moreover we define the *graded order*, $\mathcal{O}(g)$, of a function $g \in C^\infty(U)$ to be the smallest weight of the monomials that appear in any Taylor approximation of g at x_0 .

For example, let $n = 2$ and suppose that x_1 has weight 1 and x_2 has weight 2. Then the polynomial $x_1 x_2 - \frac{(x_1)^2 (x_2)^2}{6}$ has weight 6, because the two monomials composing it are $x_1 x_2$ of weight 3 and the rest of weight 6. On the other hand, $\sin(x_1 x_2) = x_1 x_2 - \frac{(x_1)^2 (x_2)^2}{6} + o((x_1)^2 (x_2)^2)$ has graded order 3.

We extend these definitions to differential operators. We say that a polynomial vector field Z is homogeneous of *weight* i if

$$\mathcal{W}(Z\varphi) = \mathcal{W}(\varphi) - i \quad \forall \text{ monomial } \varphi \text{ of weight } \mathcal{W}(\varphi).$$

In other words Z subtracts weight i to every function. Then the weight of a polynomial vector field is the smallest weight of its homogeneous components. We define the *graded order*, $\mathcal{O}(D)$, of a differential operator D by saying that

$$\mathcal{O}(D) \leq j \quad \text{if and only if} \quad \mathcal{O}(D \cdot \varphi) \geq \mathcal{O}(\varphi) - j \quad \forall \text{ polynomial } \varphi,$$

that is D subtracts at most weight j from the functions. For example the graded order of a vector field like $(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \frac{\partial}{\partial x_h}$ is

$$\mathcal{O} \left((x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \frac{\partial}{\partial x_h} \right) = w_h - \left(\sum_{j=1}^n \alpha_j w_j \right)$$

Coming back to the previous example, the graded order of a field like $\sin(x_1 x_2) \frac{\partial}{\partial x_1}$ is obtained as $1 - \mathcal{O}(\sin(x_1 x_2)) = -2$.

By means of the graded order we can give a generalization of the concept of Taylor approximation of a function up to weight h . Namely, for any $\varphi \in C^\infty(U)$ and every integer $h \geq 0$, there is a unique polynomial $\varphi_{(h)}$ of weight h such that $\mathcal{O}(\varphi - \varphi_{(h)}) \geq h$.

Definition 3.7. *The polynomial $\varphi_{(h)}$ is called the graded approximation of weight h of φ and it is the sum of the polynomials of weight less than or equal to h in the formal Taylor expansion of φ at x_0 .*

For every vector field $V \in \text{Vec}(U)$ and each integer $h \leq m$ there is a polynomial vector field $V_{(h)}$ of weight h such that $\mathcal{O}(V - V_{(h)}) \leq h - 1$.

Definition 3.8. *$V_{(h)}$ is called the graded approximation of weight h of V and it is the sum of the homogeneous vector fields of weight greater than or equal to h in the formal Taylor expansion of V at x_0 .*

Notice that, since $V_{(h)}$ is a polynomial vector field, we can consider it as defined on the whole Euclidean space \mathbb{R}^n .

We will see in the next sections how to apply this graded structure in order to underline the most important properties of the operator in (1), concerning the small time asymptotic of its fundamental solution.

4 Nilpotent approximation and the order of the dilations

In this section we apply the graded structure, that we have just developed, to define a special class of vector fields, which approximate the original one f_0, f_1, \dots, f_n and we will give an example to clarify the setting. Finally we will compute how the dilations change the volume form and we will introduce the order that will appear in the asymptotic of the heat kernel.

4.1 Nilpotent approximation

Let (x, U, w) be the graded structure around x_0 introduced in Section 3 and let f_0, f_1, \dots, f_k be the vector fields used to define the filtration of $T_{x_0}M$. Then as proved in [7] Theorem 3.1, for every $f \in L_i$, we have a bound on the graded order, namely $\mathcal{O}(f) \leq i$, where \mathcal{O} is the graded order associated to the graded structure (x, U, w) defined in Section 3.

Recall the integers l_0, l_1, \dots, l_k introduced in Section 3 to define the filtration and denote by \hat{f}_i the graded approximation of weight l_i of f_i . In other words, \hat{f}_i has weight l_i and $\mathcal{O}(f_i - \hat{f}_i) < l_i$.

Definition 4.1. The fields $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k$ are called the nilpotent approximation of f_0, f_1, \dots, f_k .

The fields \hat{f}_i , for $0 \leq i \leq k$, are polynomials, so they can be defined on \mathbb{R}^n .

We can describe more precisely the structure of the approximating fields \hat{f}_i :

- \hat{f}_0 contains the terms of weight 2; therefore every component \hat{f}_0^j of \hat{f}_0 depends only linearly on the coordinates of weight $w_j - 2$ and more then linearly on the coordinates of less weight, but does not depend on the coordinates of weight greater then or equal to $w_j - 1$, that are x_h with $h > d_{j-2}$. Moreover, since f_0 vanishes in x_0 , there are no constant terms.
- \hat{f}_i contains the terms of weight 1 for $i = 1, \dots, k$; therefore every component \hat{f}_i^j of \hat{f}_i depends only linearly on the coordinates of weight $w_j - 1$ and more then linearly on the coordinates of less weight, but do not depend on the coordinates of weight greater then or equal to w_j , that are x_h with $h > d_{j-1}$.

To make the construction more clear we end this subsection with an example, in which we will present the filtration in x_0 , the induced adapted chart and the graded structure, and we will find the related nilpotent approximation.

Example 1. Let $M = \mathbb{R}^2$, and let the number of controlled vector fields be $k = 1$. Define the vector fields

$$f_1 := \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \quad \text{and} \quad f_0 := \sin(x_1^2) \frac{\partial}{\partial x_2}$$

and recall the choice of weights $l_0 = 2$ and $l_1 = 1$. The non vanishing Lie brackets that contribute to span the tangent space in any point are given by

$$[f_1, f_0] = 2x_1 \cos(x_1^2) \frac{\partial}{\partial x_2}, \quad [f_1, [f_1, f_0]] = (2 \cos(x_1^2) - 4x_1^2 \sin(x_1^2)) \frac{\partial}{\partial x_2}.$$

Then Hörmander assumption (2) holds in any point and the filtration defined in (22) is equal to

- $L_1 = \text{span}\{f_1\}$
- $L_2 = \text{span}\{f_1, f_0\}$
- $L_3 = \text{span}\{f_1, f_0, [f_1, f_0]\}$
- $L_4 = \text{span}\{f_1, f_0, [f_1, f_0], [f_1, [f_1, f_0]]\}$

In particular the filtration in x_0 is given by

$$L_1(x_0) = L_2(x_0) = L_3(x_0) = \text{span}\left\{\frac{\partial}{\partial x_1}\right\} \quad \text{and} \quad L_4(x_0) = \mathbb{R}^2 \quad (23)$$

and the dimensions are: $d_1 = d_2 = d_3 = 1$, $d_4 = 2$.

Let us find an adapted chart to the filtration at x_0 . As one can easily see, the coordinates (x_1, x_2) are not adapted, since $(f_1)^2 \cdot x_2(x_0) = 1 \neq 0$ and the second property of the adapted chart then fails. Following the constructive proof of Lemma 3.5, one can find that the new coordinates (y_1, y_2) defined by

$$\begin{cases} y_1 = x_1 - \frac{x_1^2}{2} + x_2 \\ y_2 = -\frac{x_1^2}{2} + x_2 \end{cases}$$

give an adapted chart at x_0 . In this coordinates the two vector fields are written as

$$f_1 = \frac{\partial}{\partial y_1} \quad \text{and} \quad f_0 = \sin((y_1 - y_2)^2) \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right).$$

For $\epsilon > 0$ the dilations defined in Definition 3.6 are

$$\delta_\epsilon : (y_1, y_2) \mapsto (\epsilon y_1, \epsilon^4 y_2). \quad (24)$$

Then the weights of the coordinate functions are $\mathcal{W}(y_1) = 1$ and $\mathcal{W}(y_2) = 4$, while the weights of the coordinate vector fields are $\mathcal{W}(\frac{\partial}{\partial y_1}) = 1$ and $\mathcal{W}(\frac{\partial}{\partial y_2}) = 4$.

Finally, let us write the Taylor expansion of the two vector fields f_1, f_0 :

$$\begin{aligned} f_1 &= \frac{\partial}{\partial y_1} \\ f_0 &= (y_1^2 - 2y_1 y_2 + y_2^2 + o(|(y_1, y_2)|^2)) \frac{\partial}{\partial y_2} \end{aligned}$$

We can see that f_1 has already weight 1, while the only part of weight 2 in f_0 is $y_1^2 \frac{\partial}{\partial y_2}$. We can therefore conclude that the nilpotent approximation of f_0, f_1 is given by

$$\hat{f}_0 = y_1^2 \frac{\partial}{\partial y_2} \quad \text{and} \quad \hat{f}_1 = \frac{\partial}{\partial y_1}.$$

4.2 Order of the dilations

We will analyze here the order of the dilations, that is the order of homogeneity of the volume form under the action of the dilations. This number will be crucial to find the order of degeneracy of the fundamental solution of the operator (1).

Let us consider the dilations δ_ϵ . They were defined by introducing the notation $x = (x^1, \dots, x^m)$, where each component x^i is a vector of length $k_i = d_i - d_{i-1}$. Then we set $\delta_\epsilon(x^1, x^2, \dots, x^m) = (\epsilon x^1, \epsilon^2 x^2, \dots, \epsilon^m x^m)$. Let N be the order of homogeneity of the volume form $dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$ around the point x_0 , that is a number such that

$$(\delta_\epsilon)_*(dx_1 \wedge dx_2 \wedge \dots \wedge dx_n) = \epsilon^N dx_1 \wedge dx_2 \wedge \dots \wedge dx_n.$$

Then N is given by

$$N := \sum_{i=1}^m i \cdot k_i = \sum_{i=1}^m i (\dim L_i(x_0) - \dim L_{i-1}(x_0)). \quad (25)$$

Since this number is very important we give here some examples to understand its meaning.

Example 2 (Continuation of Example 1). As a first example we consider the one given in Example 1. We have already computed the filtration in the equations in (23), so we already know that the integers $k_i := \dim L_i - \dim L_{i-1}$ are

$$k_1 = 1 \quad k_2 = k_3 = 0 \quad \text{and} \quad k_4 = 1.$$

Therefore the order of the dilations is $N = 1 \cdot 1 + 4 \cdot 1 = 5$, as one can compute directly from the explicit expression of the dilations in (24). Notice that this number is much bigger than the dimension of the manifold.

Example 3 (Sub-Riemannian manifold). Let us assume that the operator in (1) is induced by an equiregular sub-Riemannian manifold. In other words, we consider an operator without drift field, and the vector fields f_1, \dots, f_k generate a completely non-holonomic equiregular distribution, Δ , of step m . Recall that the growth vector of the distribution is defined as the vector at any point q of the manifold given by

$$(\Delta(q), \Delta^2(q), \dots, \Delta^m(q)) \quad \text{where} \quad \Delta^{i+1} := [\Delta, \Delta^i].$$

Then the integers k_i related to the filtration are the same defined by the growth vector, i.e. $k_i = \dim(\Delta_i) - \dim(\Delta_{i-1})$, and the number N is exactly the homogeneous dimension \mathcal{Q} of the manifold. More explicitly

$$N = \mathcal{Q} = 1 \cdot k + 2 \cdot k_2 + \cdots m \cdot k_m = \sum_{i=1}^m i(\dim(\Delta_i) - \dim(\Delta_{i-1})).$$

In particular for the $n+1$ dimensional Heisenberg group the homogeneous dimension is $\mathcal{Q} = n+2$.

Example 4 (Linear case). As a last example, we consider an involutive distribution \mathcal{D} , spanned locally by k constant vector fields, and assume that the drift field is linear in a neighborhood of x_0 . Without loss of generality we can assume that

$$f_i = \frac{\partial}{\partial x_i} \quad \forall 1 \leq i \leq k \quad \text{and} \quad f_0 = \sum_{i,j=1}^n A_{ij} x_i \frac{\partial}{\partial x_j}$$

for some constants A_{ij} . Under these assumptions, the only Lie brackets different from zero are the one involving only one vector field of the distribution and the drift field. Let us call A the $n \times n$ matrix with entries equal to A_{ij} and B the $n \times k$ matrix that is the identity in the first k rows and is equal to zero in the last $n - k$ rows. Then Hörmander's condition of hypoellipticity (2) becomes Kalman's condition of controllability for linear control systems, that is the following condition on the rank of Kalman's $n \times (nk)$ matrix

$$\text{rank}[B, AB, A^2B, \dots, A^{n-1}B] = n. \quad (26)$$

The filtration is then completely determined and we have

$$L_{2i-1}(x_0) = L_{2i}(x_0) = \text{span}\{A^j B : 0 \leq j \leq i-1\}.$$

Consequently the numbers $d_{2i-1} = d_{2i}$ are determined by the rank of the Kalman's matrix in (26), where we stop the series of matrices at $A^{i-1}B$. The numbers k_j are zero if j is even, while if $j = 2i - 1$ they are the number of new linearly independent columns obtained by adding the matrix $A^{i-1}B$ to the previous one. The step of the distribution is then an odd number $2\tilde{m} - 1$ and N is equal to an odd sum of integers:

$$N = \sum_{i=1}^{\tilde{m}} (2i-1)k_{2i-1} = 1 \cdot k_1 + 3 \cdot k_3 + 5 \cdot k_5 + \cdots + (2\tilde{m}-1)k_{2\tilde{m}-1}.$$

This number is already appeared in literature, namely in Chapter 4 of [1], where Agrachev, Barilari and Rizzi compute the small time asymptotics of the cost functional associated to any ample, equiregular geodesic γ . In particular, if one computes the asymptotic of the cost functional associated to the ample, equiregular geodesics $\gamma(t) \equiv x_0$, then the trace of the principal term of the expansion is exactly the same term N . We refer to [1] for an exhaustive presentation.

5 Small time asymptotic on the diagonal

We come back now to the perturbative method explained in Section 2. Let

$$\frac{\partial \varphi}{\partial t} - f_0(\varphi) - \frac{1}{2} \sum_{i=1}^k f_i^2(\varphi) \quad \forall \varphi \in C^\infty(\mathbb{R} \times M)$$

be the differential operator (1) on $\mathbb{R}^+ \times M$ and assume that f_0, f_1, \dots, f_k satisfy Hörmander condition (2). Fix the graded structure, (U, x, w) , around x_0 , induced by the filtration \mathcal{L} introduced in Section 3 and let $p(t, x, y)$ be the fundamental solution of the partial differential equation corresponding to a volume form μ , such that $\mu = dx_1 \wedge \dots \wedge dx_n$ in U . By Remark 2.2 this assumption on μ is not restrictive for our study. For $0 < \epsilon < 1$ we dilate the space around x_0 with the dilation δ_ϵ defined in Definition 3.6, then by Proposition 2.4 the fundamental solution in U of the operator

$$\frac{\partial}{\partial t} - \epsilon^2 \left(\delta_{1/\epsilon*} f_0 + \frac{1}{2} \sum_{i=1}^k (\delta_{1/\epsilon*} f_i)^2 \right) \quad (27)$$

is

$$q_\epsilon(t, x, y) = \epsilon^N p(\epsilon^2 t, \delta_\epsilon x, \delta_\epsilon y) \quad \forall x, y \in U \quad (28)$$

where N is the order of homogeneity of the volume form μ computed in (25).

The fields f_0, f_1, \dots, f_k can be written in terms of the nilpotent approximation as

$$f_0 = \hat{f}_0 + g_0, \quad f_i = \hat{f}_i + g_i \quad 1 \leq i \leq k,$$

where g_0 and g_i are vector fields of order less than or equal to 1 and 0 respectively. Then the operator in (27) can be decomposed in a principal part, perturbed by a small operator:

$$\begin{aligned} \frac{\partial}{\partial t} - \epsilon^2 \left(\delta_{1/\epsilon*} f_0 + \frac{1}{2} \sum_{i=1}^k (\delta_{1/\epsilon*} f_i)^2 \right) &= \\ &= \frac{\partial}{\partial t} - \epsilon^2 \left(\frac{1}{\epsilon^2} \hat{f}_0 + o\left(\frac{1}{\epsilon^2}\right) + \frac{1}{2} \sum_{i=1}^k \left(\frac{1}{\epsilon} \hat{f}_i + o\left(\frac{1}{\epsilon}\right) \right)^2 \right) \\ &= \frac{\partial}{\partial t} \hat{f}_0 - \frac{1}{2} \sum_{i=1}^k \hat{f}_i^2 + o(1) \\ &= \frac{\partial}{\partial t} - L_0 - \mathcal{X}_\epsilon \end{aligned}$$

where \mathcal{X}_ϵ is an operator that goes to zero as ϵ and L_0 is the operator defined by the nilpotent approximation:

$$L_0 := \hat{f}_0 + \frac{1}{2} \sum_{i=1}^k \hat{f}_i^2.$$

To this kind of operator we can apply Duhamel's formula (19), but to this end we have to prove that there exists the fundamental solution of the principal operator $\partial/\partial t = L_0$. As we will prove now, this will follow by the property of hypoellipticity of the original operator, that are preserved by the nilpotent approximation, that defines L_0 . The same statement can be found also in the paper by Bianchini and Stefani [7].

Proposition 5.1. *Let $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k$ be the nilpotent approximation of the fields f_0, f_1, \dots, f_k defined in Definition 4.1. Then*

(i) *for every bracket Λ such that $||\Lambda|| = i$, then $(\Lambda_f - \Lambda_{\hat{f}}) \in L_{i-1}(x_0)$ and $\Lambda_{\hat{f}}(x_0) = 0$ whenever $\Lambda_f(x_0) \in L_{i-1}(x_0)$, where $||\Lambda||$ denotes the weight of the bracket Λ defined in Definition 3.2.*

(ii) *Assume $L_m(x_0) = \text{Lie}_{x_0}\{f_0, f_1, \dots, f_k\}$, then*

$$\text{Lie}_{x_0}\{f_0, f_1, \dots, f_k\} = \text{Lie}_{x_0}\{\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k\}. \quad (29)$$

Proof. Let us prove the first statement. Let (U, x) be coordinates around x_0 adapted to the filtration $\{L_i\}_i$. Then for every i ,

$$L_i(x_0) = \text{span} \left\{ \frac{\partial}{\partial x_1} \Big|_{x_0}, \dots, \frac{\partial}{\partial x_{d_i}} \Big|_{x_0} \right\}. \quad (30)$$

Let Λ be a bracket such that $\|\Lambda\| = i$, then as proved in [7] Theorem 3.1, $\mathcal{O}(\Lambda_f) \leq l_i$, where \mathcal{O} is the graded order associated to the graded structure induced by the filtration. Therefore there exist constants a_j such that

$$\Lambda_f(x_0) = \sum_{j: l_j \leq l_i} a_j \frac{\partial}{\partial x_j} \Big|_{x_0}. \quad (31)$$

Notice that if two vector fields h_1, h_2 are homogeneous of graded order respectively n_1 and n_2 , then their Lie bracket is either zero or homogeneous of order $n_1 + n_2$. Then the Lie bracket $\Lambda_{\hat{f}}$ is either zero or homogeneous of order $\|\Lambda\| = i$. Therefore by equation (31), we have

$$\Lambda_{\hat{f}}(x_0) = \sum_{j: l_j = l_i} a_j \frac{\partial}{\partial x_j} \Big|_{x_0}. \quad (32)$$

By subtracting (32) to (31), we find that $(\Lambda_f - \Lambda_{\hat{f}}) \in L_{i-1}(x_0)$, because $L_i(x_0)$ is obtained as in (30).

For the second statement, let (U, x) be as before and let $\hat{V}(x_0) \in \text{Lie}_{x_0}\{\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k\}$. Then $\hat{V}(x_0) = \Lambda_{\hat{f}}(x_0)$, for some bracket Λ with $\|\Lambda\| = j$ equal to the graded order of $\hat{V}(x_0)$. Then by expression (30), there exist α_i such that $\hat{V}(x_0) = \sum_{i: l_i = j} \alpha_i \frac{\partial}{\partial x_i}$. Since $L_j(x_0) \subset L_m(x_0) = \text{Lie}_{x_0}\{f_0, f_1, \dots, f_k\}$, we have that $\hat{V}(x_0) \in \text{Lie}_{x_0}\{f_0, f_1, \dots, f_k\}$.

We prove the other inclusion by proving that $L_i(x_0) = \text{span}\{\Lambda_{\hat{f}}(x_0) : \|\Lambda\| \leq i\}$, for every i . We prove it by induction on i .

For $i = 1$, $L_1(x_0) = \text{span}\{\Lambda_f(x_0) : \|\Lambda\| \leq 1\}$. Let $\Lambda_f(x_0) \in L_1(x_0)$, then by statement (i), $(\Lambda_f - \Lambda_{\hat{f}})(x_0) \in L_0(x_0) = \{0\}$. Then $\Lambda_f(x_0) = \Lambda_{\hat{f}}(x_0)$ and the statement is true for $i = 1$.

Assume that the statement is true for $i - 1$, then $L_i(x_0) = \text{span}\{\Lambda_f(x_0) : \|\Lambda\| \leq i\}$ and $(\Lambda_f - \Lambda_{\hat{f}})(x_0) \in L_{i-1}(x_0)$. By the induction hypothesis, there exists $g \in \text{span}\{\Lambda_{\hat{f}}(x_0) : \|\Lambda\| \leq i - 1\}$ such that

$$\Lambda_f(x_0) = \Lambda_{\hat{f}}(x_0) + g.$$

And the statement is proved also for i .

We conclude, since $\text{Lie}_{x_0}\{f_0, f_1, \dots, f_k\} = L_m(x_0) \subset \text{Lie}_{x_0}\{\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k\}$. \square

Corollary 5.2. *The operator $\partial/\partial t = L_0$ is hypoelliptic on \mathbb{R}^n .*

Proof. By Hörmander's condition of hypoellipticity we know that $\text{Lie}_{x_0}\{f_0, f_1, \dots, f_k\} = \mathbb{R}^n$. Then the hypothesis of statement (ii) of the Proposition are fulfilled and then also the nilpotent approximation is Lie bracket generating. Moreover, since $f_0(x_0) = 0 = \hat{f}_0(x_0)$, the approximation of the drift field can give some contribution in the generating process at x_0 only trough its Lie brackets with some other vector field. Then even Hörmander's condition of hypoellipticity (2) holds at x_0 for the nilpotent approximation, that is

$$\text{span}_{x_0} \left\{ \frac{\partial}{\partial t} - \hat{f}_0, \hat{f}_1, \dots, \hat{f}_k \right\} = \mathbb{R}^{n+1}.$$

Using the lower semi-continuity of the rank, we can then find a small neighborhood U of x_0 where Hörmander condition holds at any point.

Now by the homogeneity of the approximating system we know that

$$\delta_{\epsilon*}\hat{f}_0 = \epsilon^2\hat{f}_0 \quad \text{and} \quad \delta_{\epsilon*}\hat{f}_i = \epsilon\hat{f}_i \quad \forall 1 \leq i \leq k.$$

Therefore, since the differential operator commutes with the Lie brackets, we can extend Hörmander condition, which holds on a neighborhood of x_0 , to the whole Euclidean space \mathbb{R}^n and the operator $\partial/\partial_t = L_0$ is hypoelliptic on \mathbb{R}^n . \square

Remark 5.3. The assumption of $f_0(x_0) = 0$ is necessary for the proof of this corollary. Indeed if $f_0(x_0) \neq 0$ it could be that L is hypoelliptic, but L_0 is not.

For example, on \mathbb{R}^2 the fields

$$f_1 = \frac{\partial}{\partial x_1} \quad f_0 = (1 + x_1)\frac{\partial}{\partial x_2}$$

satisfy Hörmander condition (2), but this fails for their nilpotent approximation

$$\hat{f}_1 = \frac{\partial}{\partial x_1} \quad \hat{f}_0 = \frac{\partial}{\partial x_2}.$$

We can conclude that the principal part of the operator in (27) admits a well defined heat kernel, $q_0(t, x, y)$, for small time, that is given by the density function of the solution, $\xi(t)$, of the stochastic differential equation in Stratonovich form

$$\begin{aligned} d\xi &= \hat{f}_0(\xi)dt + \sum_{i=1}^k \hat{f}_i(\xi) \circ dw_i \\ \xi(0) &= x \end{aligned} \tag{33}$$

where w_i is a 1-dim Brownian motion for every $1 \leq i \leq k$. We can then apply the procedure introduced in Section 2 and we conclude by giving the asymptotic on the diagonal in x_0 of the fundamental solution $p(t, x, y)$.

Theorem 5.4. *Let f_0, f_1, \dots, f_k be smooth vector fields that satisfy Hörmander condition and (U, x, w) be a graded coordinate neighborhood induced by the filtration \mathcal{L} defined in (22) around a point x_0 , where $f_0(x_0) = 0$. Let $q_0(t, x, y)$ be the probability density function of the solution ξ_t to equation (33) with initial condition $\xi(0) = x$ and assume that $q_0(1, x_0, x_0)$ is strictly positive. Then the short time asymptotic on the diagonal of the fundamental solution, $p(t, x, y)$, of the heat operator*

$$\frac{\partial}{\partial t} - f_0 - \frac{1}{2} \sum_{i=1}^k f_i^2 \tag{34}$$

is given by

$$p(t, x_0, x_0) = \frac{q_0(1, x_0, x_0)}{t^{N/2}}(1 + o(1)), \tag{35}$$

where N is the degree of homogeneity of the volume form $dx_1 \wedge \dots \wedge dx_n$ under the action of the dilations δ_ϵ computed in (25).

Proof. By Duhamel's formula (19) the asymptotic of the fundamental solution q_ϵ to the dilated operator as ϵ tends to 0 is

$$q_\epsilon(t, x, y) = q_0(t, x, y) + q_\epsilon * \mathcal{X}_\epsilon q_0(t, x, y), \tag{36}$$

provided that the remainder term $q_\epsilon * \mathcal{X}_\epsilon q_0$ is negligible for ϵ small.

Let \mathcal{Z} be the operator $\frac{\mathcal{X}_\epsilon}{\epsilon}$. We have to prove that

$$\int_0^1 \int_U q_\epsilon(s, x_0, y) \mathcal{Z} q_0(1-s, y, x_0) dy ds$$

is bounded uniformly in ϵ . We split the integral between the integration on $0 < s < \frac{1}{2}$ and the integration on $\frac{1}{2} < s < 1$. For the first integral, since we are integrating on a bounded set where $1-s$ is far from zero, $\mathcal{Z} q_0(1-s, y, x_0)$ is uniformly bounded by a constant C . Then

$$\begin{aligned} \int_0^{1/2} \int_U q_\epsilon(s, x_0, y) \mathcal{Z} q_0(1-s, y, x_0) dy ds &\leq C \int_0^{1/2} \int_U q_\epsilon(s, x_0, y) dy ds \\ &= C \int_0^{1/2} \int_U \epsilon^N p(\epsilon^2 s, x_0, \delta_\epsilon y) dy ds \\ &\leq \frac{C}{2} \end{aligned}$$

where the last inequality follows since p has integral equal to 1 over the whole space.

For the other part of the integral, the smooth function $q_\epsilon(s, x_0, y)$ is uniformly bounded on U , since s is far from 0. Moreover, the operator \mathcal{Z} is a combination of polynomials, that are bounded on the bounded set U and of derivations of q_0 of any order. So let us control the integral

$$\int_{1/2}^1 \int_U \frac{\partial}{\partial x_i} q_0(1-s, y, x_0) dy ds$$

for every $1 \leq i \leq n$. By an integration by parts, it is equal to

$$\int_{1/2}^1 \int_{U \cap \mathbb{R}^{n-1}} q_0(1-s, y_{U_1}, x_0) - q_0(1-s, y_{U_2}, x_0) dy_1 \cdots \hat{dy}_i \cdots dy_n ds$$

where y_{U_1} and y_{U_2} are the two extremals of the set U on the line of integration in y_i , while \hat{dy}_i denotes that we are not integrating in dy_i . This integral is bounded since q_0 has integral equal to 1 on the whole space.

For the higher derivatives, let us assume by induction that for every set of indexes I , with $|I| = j$, we have proved that the integral

$$\int_{1/2}^1 \int_U \frac{\partial^I}{\partial x_I} q_0(1-s, y, x_0) dy ds$$

is bounded. Let J be a set of indexes such that $|J| = j+1$, and that it differs from I by an index i , then we can integrate by parts in the coordinate x_i the whole integral

$$\begin{aligned} \int_{1/2}^1 \int_U q_\epsilon(s, x_0, y) \frac{\partial^J}{\partial x_J} q_0(1-s, y, x_0) dy ds &= \\ &= \int_{1/2}^1 \int_{U \cap \mathbb{R}^{n-1}} q_\epsilon(s, x_0, y) \frac{\partial^I}{\partial x_I} q_0(1-s, y, x_0) \Big|_{y_{U_2}}^{y_{U_1}} dy_1 \cdots \hat{dy}_i \cdots dy_n ds - \\ &\quad - \int_{1/2}^1 \int_{U \cap \mathbb{R}^{n-1}} \int_{U \cap \mathbb{R}} \frac{\partial q_\epsilon}{\partial x_i}(s, x_0, y) \frac{\partial^I}{\partial x_I} q_0(1-s, y, x_0) dy ds \end{aligned}$$

and this is bounded by the induction hypothesis and since q_ϵ and $\frac{\partial q_\epsilon}{\partial x_i}$ are bounded away from $s = 0$.

Now let us come back to the asymptotic in (36). By the definition of q_ϵ , if in (28) we fix $t = 1$, $x = y = x_0$, and we let ϵ go to zero as $\epsilon = \sqrt{t}$, we notice that

$$p(t, x_0, x_0) = \frac{q_{\sqrt{t}}(1, x_0, x_0)}{\sqrt{t}^N}.$$

The desired small time asymptotic on the diagonal is then determined by the asymptotic (36) for q_ϵ and we find

$$p(t, x_0, x_0) = \frac{q_0(1, x_0, x_0)}{t^{N/2}}(1 + o(1)),$$

which is well defined since by hypothesis the leading term $q_0(1, x_0, x_0)$ doesn't vanish. \square

Remark 5.5. The conclusion of Theorem 5.4 holds also for the operator

$$\frac{\partial}{\partial t} - f_0 - \frac{1}{2} \sum_{i=1}^k (f_i^2 + (\operatorname{div}_\mu f_i) f_i) \quad (37)$$

where μ is any volume form on M and f_0, f_1, \dots, f_k satisfy the assumptions of Theorem 5.4.

Indeed we can treat the operator in (37) as an operator of the form

$$\frac{\partial}{\partial t} - \tilde{f}_0 - \frac{1}{2} \sum_{i=1}^k f_i^2$$

where we take $\tilde{f}_0 = f_0 + \frac{1}{2} \sum_{i=1}^k (\operatorname{div}_\mu f_i) f_i$, but in this case we could have $\tilde{f}_0(x_0) \neq 0$. However, if f_0, f_1, \dots, f_k satisfy Hörmander condition (2), then it is satisfied also by $\tilde{f}_0, f_1, \dots, f_k$. Moreover, the filtration $\tilde{\mathcal{L}}$ generated by $\tilde{f}_0, f_1, \dots, f_k$ is equal to the filtration \mathcal{L} generated by f_0, f_1, \dots, f_k , since \tilde{f}_0 differs from f_0 by a linear combination of the vector fields f_1, \dots, f_k . Then the approximating system $\hat{\tilde{f}}_0, \hat{\tilde{f}}_1, \dots, \hat{\tilde{f}}_k$ is the same as the approximating system $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k$, hence the operators in (37) and in (34) have the same principal part. The proof of Theorem 5.4 for the operator in (37) now follows as the one for the operator (34).

6 The principal operator and the associated control system

In this section we are going to investigate the conditions for the positivity of the heat kernel, $q_0(t, x, y)$, that we have introduced in the last section.

Let f_0, f_1, \dots, f_k satisfy Hörmander condition (2) and consider the principal operator

$$\frac{\partial}{\partial t} - \hat{f}_0 - \frac{1}{2} \sum_{i=1}^k \hat{f}_i^2 \quad (38)$$

defined by the approximating system of the original system of vector fields. As already pointed out, it admits a smooth fundamental solution given by the probability density, $q_0(t, x, y)$, of the process ξ_t to be at time t in the point y starting from the point x , where ξ_t is the solution of the stochastic differential equation

$$d\xi_t = \hat{f}_0(\xi_t)dt + \sum_{i=1}^k \hat{f}_i(\xi_t) \circ dw_i. \quad (39)$$

In their famous work [13] Stroock and Varadhan characterized the support of $q_0(t, x, y)$ and they showed that it is the set of reachable points from x of the following associated control problem:

$$\dot{x} = \hat{f}_0(x) + \sum_{i=1}^k u_i \hat{f}_i(x) \quad (40)$$

where $x : [0, t] \rightarrow \mathbb{R}^n$ is a curve in \mathbb{R}^n and $u = (u_1, \dots, u_k) \in L^\infty([0, t]; \mathbb{R}^k)$ are bounded controls.

Unfortunately, Stroock and Varadhan's result holds only for globally bounded vector fields, with bounded derivatives of any order. Since our vector fields are polynomial, they don't satisfy such assumptions and we can not directly apply the result of the support theorem. We will see in a moment how we can adapt their procedure to our system, but first we introduce two simple lemmas that will simplify our study.

Lemma 6.1. *Let $q_0(t, x, y)$ be the probability density of the solution to equation (39). Then for every $\epsilon > 0$, for every $t > 0$ and for all $x \in \mathbb{R}^n$ it holds*

$$q_0(t, x_0, x) = \epsilon^N q_0(\epsilon^2 t, x_0, \delta_\epsilon x).$$

In particular, we have the following equivalence

$$q_0(1, x_0, x_0) > 0 \iff q_0(t, x_0, x_0) > 0 \quad \forall t > 0. \quad (41)$$

Proof. This is a corollary of Proposition 2.4, since by definition of $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k$ we have

$$\epsilon^2 \delta_{\frac{1}{\epsilon}*} L_0 = L_0.$$

□

Lemma 6.2. *Consider the control problem (40) on \mathbb{R}^n . Let $y_1, y_2 \in \mathbb{R}^n$ and $T > 0$ be fixed and assume there exists a curve $y : [0, T] \rightarrow \mathbb{R}^n$ that satisfies the control problem (40) for some control function $u \in L^\infty$ and such that $y(0) = y_1$ and $y(T) = y_2$.*

Then for any $M > 0$ the curve $x(t) := \delta_M(y(\frac{t}{M^2}))$ is an admissible curve for the control problem defined on $[0, M^2 T]$ with control $\tilde{u}(t) := \frac{1}{M} u(t/M^2)$, that connects $x_1 := \delta_M(y_1)$ with $x_2 := \delta_M(y_2)$ in time $M^2 T$.

In particular, if the control problem (40) is controllable in a neighborhood U of x_0 in time T , then it is controllable in $\delta_M(U)$ in time $M^2 T$.

Proof. The boundary conditions are easily satisfied, since $x(0) = \delta_M(y(0)) = \delta_M(y_1) = x_1$ and $x(M^2 T) = \delta_M(y(T)) = \delta_M(y_2) = x_2$. Moreover, $x(t)$ is an admissible curve for the control problem with control $\frac{1}{M} u_i(t/M^2)$, indeed by the homogeneity of the approximating system we have

$$\begin{aligned} \dot{x}(t) &= \frac{1}{M^2} \delta_{M*}(\dot{y}(t/M^2)) \\ &= \frac{1}{M^2} \delta_{M*} \left[\hat{f}_0(y(t/M^2)) + \sum_{i=1}^k u_i(t/M^2) \hat{f}_i(y(t/M^2)) \right] \\ &= \frac{1}{M^2} \left[M^2 \hat{f}_0(y(t/M^2)) + \sum_{i=1}^k M u_i(t/M^2) \hat{f}_i(y(t/M^2)) \right] \\ &= \hat{f}_0(x(t)) + \sum_{i=1}^k \frac{1}{M} u_i(t/M^2) \hat{f}_i(x(t)). \end{aligned}$$

□

We generalize here Stroock and Varadhan's support theorem for the control problem defined by the approximating system. Indeed their theorem in [13] can be applied only to fields that are bounded and Lipschitz, with bounded derivatives of the first and second order, and does not hold for general unbounded fields. In our case, the fields are polynomial, so in general not globally bounded. However they are still very particular, because, since \hat{f}_0 has weight 2 and \hat{f}_i has weight 1, for every $1 \leq i \leq k$, every component of these fields depends only on the coordinates of less weight, i.e. $\hat{f}_i^{(j)}$ does not depend on x_j, x_{j+1}, \dots, x_n . This will allow us to modify the proof of the support theorem in order to extend it to our case.

Definition 6.3. Consider the control problem (40) and let us call $x_u(t)$ the solution corresponding to a control u . The reachable set of the control problem in time t from x is the set,

$$\mathcal{A}_t(x) := \left\{ y \in \mathbb{R}^n : \exists u \in L^\infty([0, t]; \mathbb{R}^k) \text{ such that } x_u(0) = x \text{ and } x_u(t) = y \right\}.$$

Lemma 6.4. Let X_0, X_1, \dots, X_k be smooth vector fields on \mathbb{R}^n , that satisfy Hörmander condition, and such that every j -th component $X_i^{(j)}$ of X_i , for $0 \leq i \leq k$, does not depend on the coordinates x_j, \dots, x_n , but only on the first coordinates x_1, \dots, x_{j-1} . Let ξ_t be the solution of the stochastic differential equation

$$d\xi_t = \tilde{X}_0(\xi_t)dt + \sum_{i=1}^k X_i(\xi_t)dw_t$$

where \tilde{X}_0 stands for the vector field whose j -component is given by

$$\tilde{X}_0^{(j)} = X_0^{(j)} + \frac{1}{2} \sum_{i=1}^k \sum_{l=1}^n X_i^{(l)} \frac{\partial X_i^{(j)}}{\partial x_l}.$$

Let $p(t, x, y)$ be the probability density of ξ_t to be in y at time t starting from the point x . Let $\mathcal{A}_t(x)$ be the reachable set at time t from x of the associated control problem

$$\dot{x} = X_0(x) + \sum_{i=1}^k u_i(t) X_i(x) \quad (42)$$

where $u = (u_1, \dots, u_k) \in L^\infty([0, t]; \mathbb{R}^k)$ is a control function. Then

$$\text{supp}(p(t, x, \cdot)) = \overline{\mathcal{A}_t(x)}.$$

Proof. Stroock and Varadhan have proved this theorem under the assumption that the fields are Lipschitz and globally bounded, together with their derivatives of first and second order. Following their proof in [13], we have to show that for a dense set of controls u and $\forall \epsilon > 0$

$$P_x(\|\xi_t - x_t\| < \epsilon) := P(\|\xi_t - x_t\| < \epsilon \mid \xi_0 = x) > 0,$$

where x_t is the solution of (42) starting at x . In particular let us take $\psi \in C^2(\mathbb{R}^+; \mathbb{R}^k)$, with $\psi(0) = 0$, and let x_t be the solution of (42) starting at x with control $u_i(t) := \dot{\psi}_i(t)$. Then for all $\epsilon > 0$ we show that

$$P_x(\|\xi_t - x_t\| < \epsilon \mid \|w_t - \psi_t\| < \delta) \rightarrow 1 \quad (43)$$

as $\delta \searrow 0$. Indeed

$$P_x(\|\xi_t - x_t\| < \epsilon) = P_x(\|\xi_t - x_t\| < \epsilon \mid \|w_t - \psi_t\| < \delta) \cdot P(\|w_t - \psi_t\| < \delta)$$

and $P(\|w_t - \psi_t\| < \delta) > 0$ for every $\delta > 0$.

Stroock and Varadhan proved (43) under the boundedness assumption that we don't have directly, but we will recover it by iterating a conditional probability. Indeed, notice that by our assumption the first component of every vector field, $X_i^{(1)}$, does not depend on any coordinate, so they are actually constant and they trivially satisfy Stroock and Varadhan's assumptions. Then the limit in (43) holds for the process $\|\xi_t^{(1)} - x_t^{(1)}\|$.

Moreover let us assume, by induction, that the first $j-1$ components of ξ_t live in a bounded set. Then the components $X_i^{(j)}$ are Lipschitz and bounded, together with their derivatives of any order, and we can apply Stroock and Varadhan's theorem to the j -th component of ξ_t then

$$P_x \left(\|\xi_t^{(j)} - x_t^{(j)}\| < \epsilon \mid \|w_t - \psi_t\| < \delta, \|\xi_t^{(l)} - x_t^{(l)}\| < \epsilon \forall 1 \leq l < j \right) \rightarrow 1$$

as $\delta \searrow 0$.

The proof of (43) now follows using an iterated conditional probability, indeed in general for every measurable set A_1, \dots, A_n, B , it holds

$$\begin{aligned} P \left(\bigcap_{j=1}^n A_j \mid B \right) &= P \left(\bigcap_{j=2}^n A_j \mid B \cap A_1 \right) P(A_1|B) \\ &= P \left(\bigcap_{j=3}^n A_j \mid B \cap A_1 \cap A_2 \right) P(A_2|B \cap A_1) P(A_1|B) \\ &\vdots \\ &= \prod_{j=1}^n P \left(A_j \mid B \cap \bigcap_{l=1}^{j-1} A_l \right). \end{aligned}$$

Then

$$\begin{aligned} P(\|\xi_t - x_t\| < \epsilon \mid \|w_t - \psi_t\| < \delta) &= \\ &= \prod_{j=1}^n P \left(\|\xi_t^{(j)} - x_t^{(j)}\| < \epsilon \mid \|w_t - \psi_t\| < \delta, \|\xi_t^{(l)} - x_t^{(l)}\| < \epsilon \forall 1 \leq l < j \right) \rightarrow 1 \end{aligned}$$

as $\delta \searrow 0$ and we have proved (43) in our case. \square

We are now ready to show a condition for the positivity of the fundamental solution of the approximating differential operator.

Theorem 6.5. *Let $q_0(t, x, y)$ be the fundamental solution of (38). If the reachable set $\mathcal{A}_t(x_0)$ of the associated control problem (40) is a neighborhood of x_0 for some $t > 0$, then $q_0(1, x_0, x_0) > 0$.*

Proof. By Lemma 6.1 it is enough to prove that $q_0(T, x_0, x_0) > 0$ for some $T > 0$. We will choose $T = 2t$. Moreover, by Lemma 6.2, if $\mathcal{A}_t(x_0)$ is a neighborhood of x_0 for some $t > 0$, it is a neighborhood for every $t > 0$.

Assume by contradiction that $q_0(2t, x_0, x_0) = 0$. By Chapman-Kolmogorov equation we know that

$$0 = q_0(2t, x_0, x_0) = \int_{\mathbb{R}^n} q_0(t, x_0, y) q_0(t, y, x_0) dy = \int_{\mathcal{A}_t(x_0)} q_0(t, x_0, y) q_0(t, y, x_0) dy,$$

where we can restrict the space of integration, since by Lemma 6.4, $\text{supp}(q_0(t, x_0, \cdot)) = \overline{\mathcal{A}_t(x_0)}$. Then for all $y \in \overline{\mathcal{A}_t(x_0)}$ we have $q_0(t, y, x_0) = 0$. The function $\tilde{q}(t, x, y) := q_0(t, y, x)$ is the fundamental solution of the adjoint operator to L_0 , that is

$$\frac{\partial}{\partial t} - L_0^* = \frac{\partial}{\partial t} - \hat{f}_0 - \frac{1}{2} \sum_{i=1}^k \hat{f}_i^2.$$

Then $\tilde{q}(t, x, y)$ is the probability density function of the stochastic process $\tilde{\xi}_t$ solution of the stochastic equation

$$\begin{aligned} d\tilde{\xi}_t &= -\hat{f}_0(\tilde{\xi}_t)dt + \sum_{i=1}^k \hat{f}_i(\tilde{\xi}_t) \circ dw_i(t) \\ \tilde{\xi}_0 &= x. \end{aligned}$$

By contradiction we have assumed that $\tilde{q}(2t, x_0, x_0) = 0$, then again by Chapman-Kolmogorov equation we have

$$0 = \tilde{q}(2t, x_0, x_0) = \int_{\tilde{\mathcal{A}}_t(x_0)} \tilde{q}(t, x_0, y) \tilde{q}(t, y, x_0) dy$$

where $\tilde{\mathcal{A}}_t(x_0)$ is the reachable set in time t from the point x_0 of the associated control problem

$$\dot{x} = -\hat{f}_0(x) + \sum_{i=1}^k u_i(t) \hat{f}_i(x).$$

It follows that $q_0(t, x_0, y) = \tilde{q}(t, y, x_0) = 0$ for all $y \in \tilde{\mathcal{A}}_t(x_0)$. Since the point x_0 is a stationary point for the control problem, then $x_0 \in \tilde{\mathcal{A}}_t(x_0)$, for every $t > 0$. By Krener theorem, x_0 is in the closure of $\text{int}(\tilde{\mathcal{A}}_t(x_0))$, then $\mathcal{A}_t(x_0) \cap \tilde{\mathcal{A}}_t(x_0)$ has non zero measure and $q_0(t, x_0, z) = 0$ for all z in this intersection. This is a contradiction to the support theorem.

We conclude that, if $\mathcal{A}_t(x_0)$ is a neighborhood of x_0 , then $q_0(t, x_0, x_0) > 0$ for all $t > 0$. \square

Remark 6.6. In view of Theorem 5.4 and Theorem 6.5 we can conclude the following properties about the asymptotic of the fundamental solution p of the operator (1).

- (i) if the control problem associated with the original system:

$$\dot{x} = f_0(x) + \sum_{i=1}^k u_i(t) f_i(x) \tag{44}$$

is not controllable around x_0 , that is $\mathcal{A}_t(x_0)$ is not a neighborhood of x_0 , then

$$p(t, x_0, x_0) = 0 \quad \forall t > 0.$$

Indeed, even if the support theorem can not be applied to this system, we still have that $\text{supp}(p(t, x_0, \cdot)) \subset \overline{\mathcal{A}_t(x_0)}$, therefore x_0 is on the boundary or out of the support. Since $p(t, x_0, \cdot)$ is smooth, the conclusion follows.

- (ii) if the control problem (44) is controllable in x_0 , then we study the controllability of its nilpotent approximation, defined by the fields $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k$ introduced in Section 4.

(ii.1) if the approximating control problem

$$\dot{x} = \hat{f}_0(x) + \sum_{i=1}^k u_i(t) \hat{f}_i(x) \quad (45)$$

is controllable around x_0 , then the asymptotic is given in Theorem 5.4, where we see that the fundamental solution on the diagonal in x_0 blows up for small t as the rational polynomial $\frac{a_0}{t^{N/2}}$, for a constant a_0 depending on the chosen volume and on the approximating system (45). The order N is determined by the Lie algebra generated by the fields f_0, f_1, \dots, f_k at x_0 as explained in formula (25).

(ii.2) if the approximating control problem (45) is not controllable, then either $p(t, x_0, x_0) = 0$ for every $t > 0$ (but this is not possible if the fields f_0, f_1, \dots, f_k are bounded, with bounded derivatives, as explained in the proof of Theorem 6.5), or $p(t, x_0, x_0)$ goes to infinity for small t faster than $\frac{1}{t^{N/2}}$ as shown in the following Proposition. In [5] the authors show an example, where the asymptotic goes to infinity even exponentially fast.

Proposition 6.7. *Assume that the control problem (44) is controllable around x_0 , but the approximating control problem (45) is not. If $p(t, x_0, x_0) > 0$, then $\lim_{t \searrow 0} t^N p(t, x_0, x_0) = +\infty$, where N is defined in (25).*

Proof. To show this we need to prove that all the coefficients in the asymptotic (35) vanish. The asymptotic was found introducing the fundamental solutions q_ϵ and q_0 and by using Duhamel's formula (19). By iterating it we can achieve a better approximation of the asymptotic and find the higher coefficients. Indeed, we obtain

$$q_\epsilon = q_0 + \sum_{i=1}^j q_0 (*\mathcal{X}_\epsilon q_0)^i + q_\epsilon (*\mathcal{X}_\epsilon q_0)^{j+1}$$

for every $j \in \mathbb{N}$, where $(*\mathcal{X}_\epsilon q_0)^i$ means that we iterate the convolution i times. We have to show that all the terms $q_0 (*\mathcal{X}_\epsilon q_0)^i$ vanish at the point $(1, x_0, x_0)$ for every $i \in \mathbb{N}$.

By Lemma 6.4 we know that $q_0(t, x_0, y) = 0$ for every $y \in \overline{\mathcal{A}_t(x_0)}^c$ and all $t > 0$. Let us assume by induction that

$$q_0 (*\mathcal{X}_\epsilon q_0)^i(t, x_0, y) = 0 \quad \forall y \in \overline{\mathcal{A}_t(x_0)}^c, \quad \forall t > 0. \quad (46)$$

Then for every $y \in \overline{\mathcal{A}_t(x_0)}^c$ we have

$$q_0 (*\mathcal{X}_\epsilon q_0)^{i+1}(t, x_0, y) = \int_0^t \int_{\overline{\mathcal{A}_s(x_0)}} q_0 (*\mathcal{X}_\epsilon q_0)^i(s, x_0, z) \mathcal{X}_\epsilon q_0(t-s, z, y) dz ds,$$

where the integral can be computed just on $\overline{\mathcal{A}_s(x_0)}$ by the induction hypothesis. But $q_0(t-s, z, y) \equiv 0$ on $\overline{\mathcal{A}_s(x_0)}$ by Chapman-Kolmogorov equation, then also all its derivatives vanish there. Then the integral is zero and we have proved (46) for every i .

Since q_0 is smooth and x_0 is on the boundary of $\overline{\mathcal{A}_t(x_0)}^c$, then equation (46) holds also in the point (t, x_0, x_0) for every $t > 0$, that means that all the coefficients of the asymptotic (35) are zero. \square

7 Examples

We end this paper with a study of some known examples, to understand better the meaning of Theorem 5.4.

Example 5 (Continuation of Example 1). We complete the study of Example 1. We have already computed the principal part of the dilated operator L_ϵ , that is

$$\frac{\partial}{\partial t} = \hat{f}_0 + \frac{1}{2}\hat{f}_1^2 = x_1^2 \frac{\partial}{\partial x_2} + \frac{1}{2} \frac{\partial^2}{\partial x_1^2}.$$

This operator is indeed hypoelliptic since $\hat{f}_1 = \frac{\partial}{\partial x_1}$ and $[\hat{f}_1, [\hat{f}_1, \hat{f}_0]] = 2\frac{\partial}{\partial x_2}$ span the whole tangent space in every point. Let $q_0(t, x, y)$ be the density function of the solution $\xi(t)$ of the stochastic equation (33), that in coordinates is given by

$$d\xi_1 = dw_1, \quad d\xi_2 = x_1^2 dt.$$

We can see that the second coordinate is actually deterministic and has positive derivative, so it can only increase. Consequently, if a path starts in $x_0 = (0, 0)$ the solution $\xi(t)$ will almost surely never come back to x_0 again, indeed x_0 is on the boundary of the support of $q_0(t, x_0, \cdot)$. Then the hypothesis of Theorem 5.4 that requires $q_0(1, x_0, x_0) > 0$ is not fulfilled.

Since the original control problem is controllable, this is an example of the type (ii.2).

Example 6 (Sub-Riemannian manifold: continuation of Example 3). The study of the asymptotic on the diagonal of the heat kernel on 3D contact sub-Riemannian manifolds has already been performed by Barilari in [2]. The nilpotent approximation of a sub-Riemannian manifold of dimension 3 is isometric to the Heisenberg group. Let us represent the Heisenberg group as \mathbb{R}^3 with coordinates (x, y, z) , then the approximating system can be written as

$$\hat{f}_1 = \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial z} \quad \text{and} \quad \hat{f}_2 = \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z}$$

As one can easily verify, the order of homogeneity of the volume form is given by $N = 4$, as computed also with the general formula (25).

The principal part of the dilated operator is hypoelliptic and symmetric, the associated control problem is then controllable, so there exists a well-defined symmetric heat kernel, that is positive for every $t > 0$ as seen in Theorem 6.5. The hypothesis of Theorem 5.4 are then fulfilled and we find that the asymptotic on the diagonal of the original heat kernel $p(t, q, q')$ has the following order:

$$p(t, q, q) = \frac{a_0(q) + o(1)}{\sqrt{t}^4} \quad \text{for a smooth function } a_0(q) > 0 \text{ on the manifold.}$$

This was the same order found by Barilari in [2].

This example is of the type (ii.1).

Example 7 (Ben Arous and Léandre). We consider here an example studied by Ben Arous and Léandre in [5]. Consider the space \mathbb{R}^2 with coordinates (x_1, x_2) and let

$$f_0 = x_1^a \frac{\partial}{\partial x_2}, \quad f_1 = \frac{\partial}{\partial x_1}, \quad f_2 = x_1^b \frac{\partial}{\partial x_2},$$

where a and b are positive integers. Then $x_0 = (0, x_2)$ is a stationary point of the drift field for any $x_2 \in \mathbb{R}$. The operator $L = f_0 + \frac{1}{2}(f_1^2 + f_2^2)$ satisfies even the strong Hörmander condition, i.e. the fields f_1, f_2 alone are Lie bracket generating.

In [6], it is shown the complete behaviour of the heat kernel $p(t, x, y)$ of this operator on the diagonal. We summarize here the most interesting properties for our study:

Theorem 7.1. 1. If $b \leq a + 1$, then there exists a constant $K(a, b) > 0$ such that

$$p(t, x_0, x_0) \sim \frac{K(a, b)}{\sqrt{t}^{b+2}}. \quad (47)$$

2. If $b > a + 1$ and a is even, then the fundamental solution $p(t, x_0, x_0)$ decreases with exponential velocity.

The results found in this paper agree with the statement of the theorem. Indeed, let us consider the filtration \mathcal{L} given by the vector fields f_0, f_1, \dots, f_k at a point $x_0 = (0, x_2)$. Let $m := \min\{a + 2; b + 1\}$. It is easy to verify that the subspaces of the filtration \mathcal{L} are

$$L_i(x_0) = \begin{cases} \mathbb{R} \times \{0\} & \text{if } 1 \leq i < m \\ \mathbb{R}^2 & \text{if } i = m. \end{cases}$$

Accordingly the coordinate x_1 has weight 1, while the coordinate x_2 has weight m and the order of homogeneity of the volume form is given by

$$N = 1 + m = \begin{cases} b + 2 & \text{if } b \leq a + 1 \\ a + 3 & \text{if } b \geq a + 1. \end{cases}$$

To determine the nilpotent approximation, it is convenient to divide the study in 3 cases, depending on the value of a w.r.t. b .

If $b < a + 1$, then $m = b + 1$. The nilpotent approximation is obtained by taking the Taylor expansion of the field f_0 of order 2 and the Taylor expansion of order 1 of f_1 and f_2 . Then we find that $\hat{f}_1 = f_1$, $\hat{f}_2 = f_2$ and $\hat{f}_0 = 0$. The principal part of the dilated operator is $\frac{1}{2}(\hat{f}_1^2 + \hat{f}_2^2)$, that is hypoelliptic. Then there exists a well-defined heat kernel, $q_0(t, x, y)$, and, since the associated control system is controllable, $q_0(t, x_0, x_0) > 0$ for every $t > 0$. Then the hypothesis of Theorem 5.4 are fulfilled and we find that the small time asymptotic of the fundamental solution of L has order $N/2 = \frac{b+2}{2}$, that is exactly the one given in (47).

If $b = a + 1$, then $m = b + 1$ and the nilpotent approximation is equal to the fields f_1, f_2, f_0 themselves. Since f_1, f_2 are Lie bracket generating the associated control system

$$\dot{x} = f_0 + u_1 f_1 + u_x f_2$$

is still controllable, then the heat kernel $q_0(t, x_0, x_0)$ is positive for every $t > 0$, and we obtain again the statement (47).

If $b > a + 1$, then $\hat{f}_1 = f_1$, $\hat{f}_0 = f_0$ and $\hat{f}_2 = 0$. The principal operator $\hat{f}_0 + \frac{1}{2}\hat{f}_1^2$ is still hypoelliptic, but if a is even, then the heat kernel q_0 is zero in $x_0 = (0, x_2)$ for any $t > 0$. This is because a.e. path starting from x_0 will never come back to x_0 again, since the drift f_0 makes the first coordinate increase, if x_1 becomes different from 0. Then we can not apply Theorem 5.4 and indeed Ben Arous and Léandre have shown an exponential decrease in this case.

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References

- [1] A. A. Agrachev, D. Barilari, and L. Rizzi. The curvature: a variational approach. *arXiv:1306.5318 [math.DG]*.
- [2] D. Barilari. Trace heat kernel asymptotics in 3d contact sub-riemannian geometry. *Journal of Mathematical Sciences*, 195(3):391–411, 2013.
- [3] D. Barilari, U. Boscain, and R. W. Neel. Small time heat kernel asymptotics at the sub-riemannian cut locus. *Journal of Differential Geometry*, 92(3):373–416, 2012.
- [4] G. Ben Arous. Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus. *Annales scientifiques de l'École Normale Supérieure*, 21:307–331, 1988.
- [5] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale (i). *Probability Theory and Related Fields*, 90:175–202, 1991.
- [6] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale (ii). *Probability Theory and Related Fields*, 90:377–402, 1991.
- [7] R.M. Bianchini and G. Stefani. Graded approximations and controllability along a trajectory. *SIAM J. Control and Optimization*, 28(4):903–924, July 1990.
- [8] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.
- [9] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. North-Holland Kodansha, 1989.
- [10] R. Léandre. Majoration en temps petit de la densité d'une diffusion dégénérée. *Probability Theory and Related Fields*, 74:289–294, 1987.
- [11] R. Léandre. Minoration en temps petit de la densité d'une diffusion dégénérée. *Journal of Functional Analysis*, 74:399–414, 1987.
- [12] S. Rosenberg. *The Laplacian on a Riemannian Manifold*, volume 31 of *London Mathematical Society Student Texts*. Cambridge University Press, 1997.
- [13] D. W. Stroock and S. R. S. Varadhan. On the support of diffusion processes with applications to the strong maximum principle. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, 3:333–359, 1972.
- [14] S.R.S. Varadhan. On the behavior of the fundamental solution of the heat equation with variable coefficients. *Communications on Pure and Applied Mathematics*, 20:431–455, 1967.